# Steady finite motions of a conducting liquid

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In certain cases of steady motion of a conducting fluid in a magnetic field, the primitive equations may be integrated once, yielding a second-order partial differential equation in the stream function. This equation is highly non-linear in general, but for certain choices of basic flow and magnetic fields it is tractable. Several arbitrary functions of integration have to be evaluated to make the analysis useful. This may be done in a region that remains undisturbed. A short discussion is given to suggest a procedure for deciding in a special case whether this undisturbed region is 'upstream' or 'downstream'.

## 1. Introduction

This paper was suggested by previous work of the author on the mechanics of rotating fluids (Long 1953*a*) and fluids with density stratification (Long 1953*b*). Among other things these papers showed how the primitive equations of motion can be integrated in certain cases to yield a partial differential equation in a single dependent variable analogous to the harmonic equation of potential flow. The procedure used to do this also works in cases of conducting fluids in magnetic fields. We will show this in some detail for the axisymmetric case in the following section. The extension to the plane case is similar and will not be discussed here.

## 2. Axisymmetric flow

Consider the steady flow of a frictionless, incompressible, conducting fluid of infinite conductivity. If, as is usual, we neglect displacement currents we have the equations (Cowling 1957)

$$-\mathbf{v}\times(\nabla\times\mathbf{v}) = -\nabla\left(\frac{p}{\rho}+\tfrac{1}{2}q^2+\chi\right)-\mathbf{h}\times(\nabla\times\mathbf{h}), \tag{1}$$

$$\nabla.\mathbf{v} = 0, \tag{2}$$

$$\nabla.\mathbf{h} = 0, \tag{3}$$

$$\nabla\times(\mathbf{v}\times\mathbf{h}) = 0, \tag{4}$$

where $\mathbf{v}$ is the fluid velocity vector, $p$ is fluid pressure, $\rho$ is the uniform density, $q = |\mathbf{v}|$ is the speed and $\chi$ is the potential of other body forces. We have written $\mathbf{h} = \mathbf{H}\sqrt{(\mu/4\pi\rho)}$, where $\mathbf{H}$ is the magnetic field and $\mu$ is the permeability.

We adopt the co-ordinate system of figure 1 and the following additional assumptions:

(1) There is axial symmetry, i.e. all scalars are independent of $\theta$. In particular, if the velocity and magnetic fields are

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \tag{5}$$

$$\mathbf{h} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}, \tag{6}$$

the components depend only on $r$ and $z$.

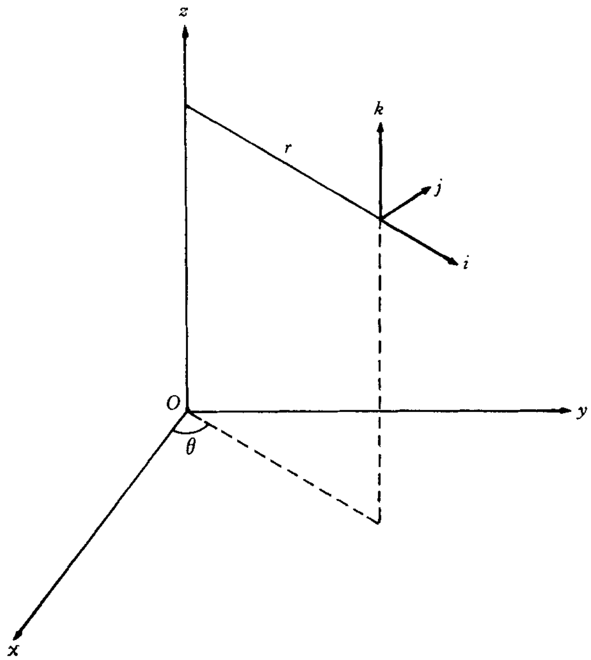


FIGURE 1. Co-ordinate system.

(2) The velocity and magnetic fields are assumed known either at $z \to -\infty$ *or* at $z \to +\infty$. The components $u$ and $f$ of these 'undisturbed fields' are assumed to be zero and, for simplicity, the remaining components depend only on distance from the axis. The question of whether an undisturbed region exists, and if so whether it is 'upstream' or 'downstream' will be discussed only with reference to a special case in the next section.

Equations (2) and (3) can be integrated by introducing two scalar functions $\psi(r, z)$ and $\Lambda(r, z)$, such that

$$ur = -\psi_z, \quad wr = \psi_r, \tag{7}$$

$$fr = -\Lambda_z, \quad hr = \Lambda_r. \tag{8}$$

The three equations in (4) are

$$\frac{\partial}{\partial z}(uh - wf) = 0, \tag{9}$$

$$\frac{\partial}{\partial r}(uhr - wfr) = 0, \tag{10}$$

$$\frac{\partial}{\partial r}(vf - ug) + \frac{\partial}{\partial z}(vh - wg) = 0. \tag{11}$$

Equations (9) and (10) have solution $r(uh-wf) = \text{const}$. The constant is zero, however, since $u$ and $f$ are zero in the undisturbed region. Introducing (8) into $uh-wf = 0$, we find

$$\frac{d\Lambda}{dt} = 0, \tag{12}$$

where the material derivative is

$$\frac{d}{dt} = u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}. \tag{13}$$

Equation (12) may also be written

$$-\psi_z\Lambda_r+\psi_r\Lambda_z = 0, \tag{14}$$

which has the integral

$$\Lambda = \Lambda(\psi). \tag{15}$$

Thus, $\Lambda$ is constant on the material surfaces $\psi = \text{const}$. Since the latter are stream surfaces, this expresses the well-known principle that magnetic lines move with the fluid in the ideal case of this paper.

Equation (11) may also be integrated. With use of (7) and (8) it takes the form

$$\frac{\partial}{\partial r}\left[\psi_z\frac{g}{r}-\Lambda_z\frac{v}{r}\right]-\frac{\partial}{\partial z}\left[\psi_r\frac{g}{r}-\Lambda_r\frac{v}{r}\right] = 0,$$

or

$$\frac{\partial}{\partial r}\left[\psi_z\left(\frac{g}{r}-\Lambda'\frac{v}{r}\right)\right]-\frac{\partial}{\partial z}\left[\psi_r\left(\frac{g}{r}-\Lambda'\frac{v}{r}\right)\right] = 0, \tag{16}$$

if we use (15) and write $\Lambda' = d\Lambda/d\psi$. Equation (16) is

$$\frac{d}{dt}\left[\frac{g-\Lambda'v}{r}\right] = 0,$$

so that

$$\frac{g-\Lambda'v}{r} = K(\psi), \tag{17}$$

where $K(\psi)$ is an arbitrary function.

Another conserved quantity can be found from the **j**-equation in (1). Since $p/\rho+\frac{1}{2}q^2+\chi$ is independent of $\theta$, this equation is

$$u\left(v_r+\frac{v}{r}\right)+wv_z-f\left(g_r+\frac{g}{r}\right)-hg_z = 0. \tag{18}$$

The same procedure that led to (17) permits us to integrate this equation. We get

$$vr-\Lambda'gr = L(\psi). \tag{19}$$

The final conservation equation is found by cross-differentiating the remaining two equations in (1) to eliminate $p/\rho+\frac{1}{2}q^2+\chi$. This vorticity equation is

$$\frac{\partial}{\partial z}\left[-\frac{v}{r}\frac{\partial}{\partial r}(vr)+\frac{g}{r}\frac{\partial}{\partial r}(gr)+w(u_z-w_r)-h(f_z-h_r)\right]$$
$$-\frac{\partial}{\partial r}\left[-\frac{v}{r}\frac{\partial}{\partial z}(vr)+\frac{g}{r}\frac{\partial}{\partial z}(gr)-u(u_z-w_r)+f(f_z-h_r)\right] = 0, \tag{20}$$

or

$$r\frac{d}{dt}\left[\frac{(u_z-w_r)-\Lambda'(f_z-h_r)}{r}\right]-\frac{1}{r^3}\frac{\partial}{\partial z}(vr)^2+\frac{1}{r^3}\frac{\partial}{\partial z}(gr)^2 = 0. \tag{21}$$

Equations (17) and (19) give us

$$(vr)^2 - (gr)^2 = A - Br^4, \tag{22}$$

where $A$ and $B$ are functions of $\psi$:

$$A = \frac{L^2}{1-\Lambda'^2}, \quad B = \frac{K^2}{1-\Lambda'^2}. \tag{23}$$

Using (22), we get

$$-\frac{1}{r^4}\frac{\partial}{\partial z}[(vr)^2 - (gr)^2] = \frac{A'u}{r^3} - B'ur$$

$$= \frac{A'}{r^3}\frac{dr}{dt} - B'r\frac{dr}{dt} = -\frac{d}{dt}\left(\frac{A'}{2r^2} + \frac{B'r^2}{2}\right), \tag{24}$$

and combining (21) and (24), we get

$$-\frac{u_z - w_r - \Lambda'(f_z - h_r)}{r} + \frac{A'}{2r^2} + \frac{B'r^2}{2} = M(\psi). \tag{25}$$

Introducing the stream function $\psi$, we now get

$$\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r - \frac{\Lambda'\Lambda''}{(1-\Lambda'^2)}[(\psi_z)^2 + (\psi_r)^2] + \frac{A'}{2(1-\Lambda'^2)} + \frac{B'r^4}{2(1-\Lambda'^2)} = \frac{Mr^2}{(1-\Lambda'^2)}. \tag{26}$$

This is a non-linear differential equation in $\psi$. The functions $\Lambda(\psi)$, $L(\psi)$, $K(\psi)$, $M(\psi)$ are assumed known ($A$ and $B$ may then be obtained from (23)).

According to our previous assumptions the velocity and magnetic fields in the undisturbed region may be written

$$\mathbf{v}_0 = v_0(r_0)\,\mathbf{j} + w_0(r_0)\,\mathbf{k}, \tag{27}$$

$$\mathbf{h}_0 = g_0(r_0)\,\mathbf{j} + h_0(r_0)\,\mathbf{k}, \tag{28}$$

where the components are functions of $r_0(r,z)$, the distance of the stream surface passing through $(r,\theta,z)$ from the axis of symmetry in the undisturbed region. The relation between $\psi$ and the Lagrangian variable $r_0$ is obtained by integrating

$$w_0 r_0 = \frac{d\psi}{dr_0}, \tag{29}$$

or

$$\psi = \int_0^{r_0} w_0 r_0 \, dr_0. \tag{30}$$

We may now evaluate all unknown functions of $\psi$ in terms of the known functions of $r_0$ in (27) and (28). From (8) we get

$$\Lambda = \int_0^{r_0} h_0 r_0 \, dr_0, \tag{31}$$

$$\Lambda' = \frac{h_0}{w_0}. \tag{32}$$

Equations (17), (19) and (25) show that

$$K = \frac{g_0}{r_0} - \frac{h_0}{w_0}\frac{v_0}{r_0}, \tag{33}$$

$$L = v_0 r_0 - \frac{h_0}{w_0} g_0 r_0, \tag{34}$$

$$M = \frac{1}{r_0}\frac{dw_0}{dr_0}(1-\Lambda'^2) - \Lambda'\Lambda'' w_0^2 + \frac{A'}{2r_0^2} + \frac{B' r_0^2}{2}. \tag{35}$$

These functions and $A'$, $B'$ become known functions of $\psi$ if we eliminate $r_0$ by using (30).

It is not the purpose of this note to develop applications of the equation (26), but we may point out certain cases in which the equation is tractable:

(1) If the undisturbed conditions are $v_0 = 0$, $w_0 = \text{const.}$, $g_0 = 0$, $h_0 = \text{const.}$, we have the well-known result that the flow is irrotational; thus

$$\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r = 0. \tag{36}$$

The magnetic field then does not affect the motion.

(2) In the case $v_0 = \Omega r_0$ (solid rotation), $w_0 = \text{const.}$, $g_0 = 0$, $h_0 = \text{const.}$, we have

$$\Lambda = \tfrac{1}{2}h_0 r_0^2, \quad \psi = \tfrac{1}{2}w_0 r_0^2, \quad K = -\frac{h_0\Omega}{w_0}, \quad L = \Omega r_0^2,$$

so that

$$A = \frac{4\Omega^2\psi^2}{w_0^2[1-(h_0^2/w_0^2)]},$$

$$B = \frac{h_0^2\Omega^2}{w_0^2[1-(h_0^2/w_0^2)]}.$$

Equation (26) is

$$\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r + \sigma^2\psi = \tfrac{1}{2}\sigma^2 w_0 r^2 \quad \left(\sigma = \frac{2\Omega}{w_0[1-(h_0^2/w_0^2)]}\right), \tag{37}$$

or, in terms of the perturbed streamfunction $\psi'$ (i.e. $\psi = \frac{1}{2}w_0 r^2 + \psi'$), it takes the neater form

$$\psi'_{zz} + \psi'_{rr} - \frac{1}{r}\psi'_r + \sigma^2\psi' = 0. \tag{38}$$

This is the same as the equation derived by the author for the non-conducting case ($h_0 = 0$). Solutions of interest may be found in ways similar to those in two papers of Long (1955, 1956).

(3) A number of other cases in which equation (26) is linear can be found by a procedure similar to that in a recent paper (Long 1958). This approach will not be developed here.

## 3. The undisturbed region

If we could have included dissipation in our discussion, we could be sure that the magnetic and flow fields would be undisturbed at sufficient distances from the source of the disturbance. Without dissipation, however, we will frequently have a situation in which steady perturbations can exist at indefinitely great distances from the source of the disturbance. In subcritical flow of water over an obstacle in a channel, for example, the free surface downstream to infinity is in steady wave motion (Lamb 1932). At sufficiently great distances upstream there is no disturbance. The steady-state theory is incomplete in such cases since the mathematical problem is indeterminate.

Stoker (1953) has shown that in the water-wave case the disappearance of upstream waves occurs even without dissipation if the flow problem is solved from the initial state of rest. On the other hand, Rayleigh (see Lamb 1932) found that indeterminacy of this kind can be removed by introducing a small amount of friction in an artificial way. As the coefficient of friction tends to zero the solution tends to that obtained by the approach of Stoker or by arbitrarily superimposing solutions to wipe out upstream waves. The author has verified that Rayleigh's approach is effective in a case similar to the one in this paper (Long 1955).

We can obtain definite results in the model mentioned in the last section, i.e. a stream of liquid moving at a uniform speed $w_0$ parallel to the axis, rotating with constant angular velocity $\Omega$, and under a uniform axial magnetic field $h_0$. If the disturbance is not too large we may suppose that the problem of the undisturbed region may be decided on the basis of linear theory, namely that upstream or downstream conditions will be undisturbed if no energy from the source of disturbance (in the vicinity of $z = 0$) can reach the steady waves which may exist. In the linear case the energy propagation will be at the speed of the group velocity. On the other hand, for large disturbances we recognize that effects that change the basic velocity and magnetic fields may propagate indefinitely in the direction of the assumed undisturbed region. The problem as originally posed would then be overdetermined mathematically. This occurs in the case mentioned above of water flow over an obstacle. If the flow is slow and the obstacle large, a 'blocking' wave propagates upstream, raising the water level and making it impossible to assume that upstream is undisturbed.

The blocking problem is discussed at length in Long (1955) and will not be examined here. The case of small or moderate axisymmetric disturbances leads to a simple and interesting conclusion. If we perturb the basic flow and magnetic fields slightly, we will obtain a spectrum of waves moving in the upstream and downstream directions. If we confine the system to a circular tube of arbitrary radius $b$, the waves will move at speeds given by (Long 1956)

$$\lambda^2 = \frac{4\pi^2}{[\sigma^2 - (z_n^2/b^2)]}, \tag{39}$$

where $\lambda$ is wavelength, $z_n$ are the zeros of the Bessel function $J_1(z)$ and $\sigma^2$ is now

$$\sigma^2 = \frac{4\Omega^2 c^2}{(c^2 - h_0^2)^2}. \tag{40}$$

The wave-speed or phase velocity is $c$. These are infinitesimal waves and may be superimposed. The group velocity $c_g$ is

$$\frac{c_g - c}{c} = -\frac{\lambda^2}{c^2}\frac{dc^2}{d\lambda^2}.$$

Solving (39) the phase velocity is

$$c^2 = h_0^2 + \frac{2\Omega^2}{(4\pi^2/\lambda^2)+(z_n^2/b^2)}\left[1 \pm \sqrt{\left\{1+\frac{h_0^2}{\Omega^2}\left(\frac{4\pi^2}{\lambda^2}+\frac{z_n^2}{b^2}\right)\right\}}\right], \tag{41}$$

so that

$$\frac{c_g - c}{c} = -\frac{8\Omega^2\pi^2}{\lambda^2 c^2\left(\frac{4\pi^2}{\lambda^2}+\frac{z_n^2}{b^2}\right)^2}\left[1 \pm \frac{1+\frac{h_0^2}{2\Omega^2}\left(\frac{4\pi^2}{\lambda^2}+\frac{z_n^2}{b^2}\right)}{\sqrt{\left\{1+\frac{h_0^2}{\Omega^2}\left(\frac{4\pi^2}{\lambda^2}+\frac{z_n^2}{b^2}\right)\right\}}}\right].$$

Using (41), we obtain

$$\frac{c_g - c}{c} = \mp \frac{4\pi^2/\lambda^2}{\left(\frac{4\pi^2}{\lambda^2}+\frac{z_n^2}{b^2}\right)\sqrt{\left\{1+\frac{h_0^2}{\Omega^2}\left(\frac{4\pi^2}{\lambda^2}+\frac{z_n^2}{b^2}\right)\right\}}}. \tag{42}$$

Comparing this with (41) it is seen that waves with speeds $c^2 < h_0^2$ have a group velocity greater than the phase velocity, while those with speeds $c^2 > h_0^2$ have a lower group velocity. In the steady-state problem, if $w_0 > h_0$, waves of the second kind can remain at rest against the current, and these will be found downstream. The undisturbed region will be upstream. However, if $w_0 < h_0$, the standing waves will be upstream and the undisturbed region will be downstream.

We see from (41) that there is both a maximum and minimum wave-speed. If the oncoming stream has a speed outside of these limits, no wave can exist and we would expect the disturbed motion to die out at $z = \pm\infty$. As the current approaches infinity or zero, (37) shows that the motion approaches potential flow.

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